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ON A REGULAR SOLVABILITY OF BOUNDARY VALUE PROBLEM FOR A CLASS OF FOURTH ORDER OPERATOR-DIFFERENTIAL EQUATIONS

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In this paper we present sufficient conditions for a regular solvability of boundary value problems for a class of operator-differential equations of fourth order, considered on the semi-axis. These conditions are expressed only by operator coefficients of the investigated equation. Along with this the exact values of the norms of intermediate derivatives are found with respect to the norm of the operator generated by the principal part of the equation in defined subspaces of Sobolev type. We note that they are closely connected with the solvability conditions and the principal part of the investigated equation has multiple characteristics.

Key words: operator-differential equation, self-adjoint operator, Hilbert space, regular

Consider the operator-differential equation of the form

$$\left(\frac{d}{dt}-A\right)^3\left(\frac{d}{dt}+A\right)u(t)+\sum_{j=1}^{3}A_j\frac{d^{4-j}u(t)}{dt^{4-j}}=f(t),\quad t\in R_+=[0,+\infty),\quad (1)$$

with one of the boundary conditions

$$u(0)=0 \quad (2)$$

or

$$\frac{du(0)}{dt}=0 \quad (3)$$

or

$$\frac{d^2u(0)}{dt^2}=0 \quad (4)$$

or

$$\frac{d^3u(0)}{dt^3}=0, \quad (5)$$

where A is a self-adjoint positive definite operator, A_1, A_2, A_3 are linear operators in a separable Hilbert space H, $f(t)$ and $u(t)$ are H–valued functions and derivatives are understood in the sense of the theory of distributions.

We introduce the following functional Hilbert spaces (see [1, 2]):

$$L_2(R_+; H) = \left\{ f(t) : \|f\|^2_{L_2(R_+;H)} = \int_0^{+\infty} \|f(t)\|^2_H \, dt < +\infty \right\},$$

$$W_2^4(R_+; H) = \left\{ u(t) : \|u\|^2_{W_2^4(R_+;H)} = \int_0^{+\infty} \left(\left\| \frac{d^4 u(t)}{dt^4} \right\|^2_H + \left\| A^4 u(t) \right\|^2_H \right) dt < +\infty \right\},$$

$$W_2^4(R_+; H; \{s\}) = \left\{ u(t) : u(t) \in W_2^4(R_+; H), \frac{d^s u(0)}{dt^s} = 0 \right\},$$

where s is a fixed integer, which takes only one of the following values: $s = 0$; $s = 1$; $s = 2$; $s = 3$.

Definition 1. *If the function* $u(t)$ *of the space* $W_2^4(R_+; H)$ *satisfies the equation (1) almost everywhere in* R_+*, then it will be called a regular solution of equation (1).*

Definition 2. *If for any* $f(t) \in L_2(R_+; H)$ *there exists a regular solution of the equation (1), which satisfies the boundary condition (2) in the following sense*

$$\lim_{t \to 0} \left\| A^{7/2} u(t) \right\|_H = 0,$$

and has the inequality

$$\|u\|_{W_2^4(R_+;H)} \le const \|f\|_{L_2(R_+;H)},$$

then we will say that the problem (1), (2) regularly solvable.

Similarly, we define a regular solvability of boundary-value problems (1), (3); (1), (4) and (1), (5) with only the difference that the boundary conditions (3), (4) and (5) hold in the sense $\lim_{t \to 0} \left\| A^{5/2} \frac{du(t)}{dt} \right\|_H = 0$, $\lim_{t \to 0} \left\| A^{3/2} \frac{d^2 u(t)}{dt^2} \right\|_H = 0$ and $\lim_{t \to 0} \left\| A^{1/2} \frac{d^3 u(t)}{dt^3} \right\|_H = 0$, respectively.

In this paper, we present sufficient conditions for a regular solvability of boundary-value problems (1), (2); (1), (3); (1), (4) and (1), (5). These conditions are expressed in operator coefficients of equation (1). We note that in the paper also the indicated exact values of the norms of operators

$$A^{4-j}\frac{d^j}{dt^j}:W_2^4\left(R_+;H;\{s\}\right)\to L_2\left(R_+;H\right),\ j=1,2,3,$$

relative to norms of an operator $P_0^{(s)}$ acting from the space $W_2^4\left(R_+;H;\{s\}\right)$ into the space $L_2\left(R_+;H\right)$ as follows:

$$P_0^{(s)}u(t)\equiv\left(\frac{d}{dt}-A\right)^3\left(\frac{d}{dt}+A\right)u(t),\ u(t)\in W_2^4\left(R_+;H;\{s\}\right).$$

These values of the norms are closely connected with the solvability conditions.

Questions of solvability of boundary value problems for various classes of operator-differential equations of the higher order were studied, for example, in [3-10] (see also references therein). But in all these studies, the principal part of equations differs from the case, which is considered in our work. Due to the limited size of the paper, we mainly formulate approvals and will plan their proofs briefly.

We have the following:

Theorem 1. *The operator* $P_0^{(s)}$ *is an isomorphism from the space* $W_2^4\left(R_+;H;\{s\}\right)$ *onto the space* $L_2\left(R_+;H\right)$.

The proof of this theorem is essentially resort to Fourier transform and the Banach theorem on the inverse operator.

Theorem 1 implies that the norm $\left\|P_0^{(s)}\right\|_{L_2(R_+;H)}$ is equivalent to the original norm $\|u\|_{W_2^4(R_+;H)}$ in the space $W_2^4\left(R_+;H;\{s\}\right)$. And since the operators of the intermediate derivatives

$$A^{4-j}\frac{d^j}{dt^j}:W_2^4\left(R_+;H;\{s\}\right)\to L_2\left(R_+;H\right),\ j=1,2,3$$

are continuous [2], then the following numbers are finite:

$$n_j^{(s)}=\sup_{0\neq u\in W_2^4(R_+;H;\{s\})}\left\|A^{4-j}\frac{d^ju}{dt^j}\right\|_{L_2(R_+;H)}\left\|P_0^{(s)}u\right\|_{L_2(R_+;H)}^{-1},\ j=1,2,3.$$

Theorem 2. $n_j^{(s)}$ *equal to the following numbers:*

$$n_1^{(0)} = n_3^{(3)} = \alpha^{-1/2},\ n_2^{(0)} = n_2^{(3)} = \beta_0^{-1/2},\ n_3^{(0)} = n_1^{(1)} = n_3^{(2)} = n_1^{(3)} = \frac{3\sqrt{3}}{16},$$

$$n_2^{(1)} = n_2^{(2)} = \frac{1}{2\sqrt{3}},\ n_3^{(1)} = n_1^{(2)} = \gamma^{-1/2},$$

where

$$\alpha = \frac{4}{3^{4/3}\left(9+\sqrt{57}\right)}\left[2\cdot 3^{7/3} + \left(9+\sqrt{57}\right)^{5/3} + 4\cdot 3^{2/3}\cdot\left(9+\sqrt{57}\right)^{1/3}\right],$$

β_0 *is a solution of the equation* $\beta^3 + 2\beta^2 - 39\beta - 140 = 0$ *in the interval* $(0,16)$,

$$\gamma = \frac{1}{4}\gamma_0^4\left(1-\frac{1}{16}\gamma_0^2\right),\ \gamma_0 = \frac{2}{9}\left[\left(107+9\sqrt{129}\right)^{1/3} + \frac{10}{\left(107+9\sqrt{129}\right)^{1/3}} + 8\right].$$

In finding the numbers $n_j^{(s)}$, we use the method of [4, 5]. In this case the factorization subjected to the same as in [9], the polynomial operator pencils of the eighth-order

$$P_j(\lambda;\beta;A) = (\lambda^2 E - A^2)^4 - \beta(i\lambda)^{2j} A^{8-2j},\ j = 1,2,3,$$

depend on the parameter $\beta \in \left[0,\ a_j^{-1}\right)$, $j = 1,2,3$, where $a_j = \frac{1}{256} j^j (4-j)^{4-j}$, $j = 1,2,3$, and E is the identity operator.

We denote by $P_1^{(s)}$ an operator, acting from the space $W_2^4(R_+;H;\{s\})$ into the space $L_2(R_+;H)$ as follows:

$$P_1^{(s)}u(t) \equiv \sum_{j=1}^{3} A_j \frac{d^{4-j}u(t)}{dt^{4-j}},\ u(t) \in W_2^4(R_+;H;\{s\}).$$

If we take into account the theorem of intermediate derivatives [2], we can easily prove the following

Lemma. *The operator* $P_1^{(s)}$ *is bounded from* $W_2^4(R_+;H;\{s\})$ *into* $L_2(R_+;H)$*, using the condition that the operators* $A_j A^{-j}$, $j = 1,2,3$ *are bounded in* H.

Using the above results, we can formulate the main result of the regular solvability of boundary-value problems (1), (2); (1), (3); (1), (4) and (1), (5).

The following theorem holds.

Theorem 3. *Let the operators* $A_j A^{-j}$, $j = 1,2,3$ *be bounded in* H *and the following inequality be satisfied*

$$\sum_{j=1}^{3} n_j^{(s)} \left\| A_{4-j} A^{-(4-j)} \right\|_{H \to H} < 1,$$

where the numbers $n_j^{(s)}$, $j = 1,2,3$ *(*s *is a fixed integer, which takes only one of the following values:* $s = 0$; $s = 1$; $s = 2$; $s = 3$*) which are defined in Theorem 2. Then each of the boundary value problems (1), (2); (1), (3); (1), (4) and (1), (5) is regularly solvable.*

The proof rests on the fact that under the conditions of Theorem 3 for all functions $u(t) \in W_2^4(R_+; H; \{s\})$ holds the inequality

$$c_1 \|u\|_{W_2^4(R_+;H)} \le \left\| P_0^{(s)} u + \tau P_1^{(s)} u \right\|_{L_2(R_+;H)} \le c_2 \|u\|_{W_2^4(R_+;H)}, \qquad (6)$$

where $\tau \in [0,1]$. Here the positive numbers c_1 and c_2 do not depend on the function $u(t)$ and on the parameter τ. Using (6) and applying the method of continuation of a parameter, it is proved that the equation

$$P_0^{(s)} u(t) + P_1^{(s)} u(t) = f(t)$$

has a unique solution in the space $W_2^4(R_+; H; \{s\})$ for all $f(t) \in L_2(R_+; H)$.

Remark 1. *We note that similarly the case is investigated when in the perturbed part of the equation (1) the operator coefficients are variables, i.e.* $A_j(t)$, $j = 1,2,3$ *are linear operators, defined for almost all* $t \in R_+$.

Remark 2. *In this paper the indicated results of solvability allow in combination with the method of [3] to conduct researches related to the spectral problems of the operator pencil*

$$P(\lambda) = (\lambda E - A)^3 (\lambda E + A) - \sum_{j=1}^{3} \lambda^{4-j} A_j .$$

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BİR SİNİF DÖRDTƏRTİBLİ OPERATOR-DİFERENSİAL TƏNLİKLƏR ÜÇÜN SƏRHƏD MƏSƏLƏLƏRİNİN REQULYAR HƏLL OLUNMASI HAQQINDA

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XÜLASƏ

İşdə yarımoxda baxılan bir sinif dördtərtibli operator-diferensial tənliklər üçün sərhəd məsələlərinin requlyar həll olunmasının kafi şərtləri gətirilmişdir. Bu şərtlər tədqiq olunan tənliyin yalnız operator əmsalları ilə ifadə olunubdur. Həmçinin Sobolev tipli müəyyən altfəzalarda baxılan tənliyin baş hissəsi ilə doğuran operatorun normasına nəzərən aralıq törəmə operatorlarının normalarının dəqiq qiymətləri tapılmışdır. Qeyd edək ki, bu qiymətlər həll olunma şərtləri ilə sıx əlaqəlidir, öyrənilən tənliyin baş hissəsi isə təkrarlanan xarakteristikaya malikdir.

Açar sözlər: operator-diferensial tənlik, öz-özünə qoşma operator, Hilbert fəzası, requlyar həll olunma.

О РЕГУЛЯРНОЙ РАЗРЕШИМОСТИ КРАЕВЫХ ЗАДАЧ ДЛЯ ОДНОГО КЛАССА ОПЕРАТОРНО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ЧЕТВЕРТОГО ПОРЯДКА

А.С.МУХАМЕД

РЕЗЮМЕ

В данной работе приведены достаточные условия регулярной разрешимости краевых задач для одного класса операторно-дифференциальных уравнений четвертого порядка, рассматриваемых на полуоси. Эти условия выражены лишь операторными коэффициентами исследуемого уравнения. При этом найдены точные значения норм операторов промежуточных производных относительно нормы оператора, порожденного главной частью данного уравнения, в определенных подпространствах типа Соболева. Отметим, что они имеют тесную связь с условиями разрешимости, а главная часть изучаемого уравнения имеет кратную характеристику.

Ключевые слова: оператор-дифференциальное уравнение, самосопряженный оператор, гильбертово пространство, регулярная разрешимость.

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